Generic Properties of Laplace Eigenfunctions in the Presence of Torus Actions

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Collaborators

Donato Cianci, Chris Judge, Samuel Lin & S–, *Spectral multiplicity and nodal sets for generic torus-invariant metrics*, arXiv:220714405

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Outline

Outline

Introduction

- M is a closed and connected manifold
- $\mathscr{R}^\ell(M)$ space of C^ℓ-Riemannian metrics, $\ell = 0, 1, \ldots, +\infty$.
- For $g \in \mathscr{R}^\ell(M)$, with $\ell \geq 2$, define the associated Laplace operator $\Delta_g = -\operatorname{div}_g \circ \nabla_g$:

$$\Delta_g : H^k(M; \mathbb{R}) \to H^{k-2}(M; \mathbb{R}).$$

- $\mathrm{Spec}_\Delta(M, g)$ will denote the spectrum of (M, g):

$$\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \nearrow +\infty$$

  the sequence of Δ_g-eigenvalues repeated according to their multiplicity.
- $\{\phi_k\}_{k=0}^\infty$ will denote an orthonormal basis of Δ_g-eigenfunctions:

$$\Delta_g \phi_k = \lambda_k \phi_k.$$

- $L^2(M)_\lambda$ is the Δ_g-eigenspace corresponding to λ.

Introduction

Inverse Spectral Problem: Understand the mutual influences between the geometry of (M, g) and spectral data (e.g., eigenvalues, eigenfunctions, etc.) associated to (M, g).

Why study Laplace eigenfunctions and eigenvalues?

- The wave equation

$$\left(\frac{\partial^2}{\partial^2 t} + \Delta_g\right) u = 0.$$

- General solution to wave equation

$$u(x,t) = \sum_{k=0}^{\infty} \underbrace{\left(a_k \cos(\sqrt{\lambda_k}\, t) + b_k \sin(\sqrt{\lambda_k}\, t)\right) \phi_k(x)}_{u_k(x,t)}$$

- $f_k = \frac{\sqrt{\lambda_k}}{2\pi}$ are the natural frequencies at which the manifold will vibrate.

Introduction

Why study Laplace eigenfunctions and eigenvalues?

- Schrödinger's equation

$$\left(i\hbar\frac{\partial}{\partial t} + \hbar^2\Delta_g\right) u(x,t) = 0$$

  governs motion of a free particle
- Via Schrödinger's equation, Laplace eigenfunctions give rise to probability measures on M

$$\Delta_g\phi = \lambda\phi \longmapsto \mu_\phi(A) \equiv \int_A \|\phi(x)\|^2 \, d\nu_g$$

  $\mu_\phi(A)$ is the probability of finding a free particle in A
- Bohr's correspondence principle suggests that as $\lambda \to \infty$, the measures μ_{ϕ_λ} should reflect aspects of the geodesic flow (i.e., classical mechanics).

Introduction

Basic Questions. In light of their importance, it is natural to wonder:

- To what degree can the spectrum be computed explicitly?
- To what degree can we find explicit Laplace eigenfunctions?

Introduction

Example (Flat Tori)

Consider the flat torus $\mathbb{T}^n = \mathbb{R}^n/\Gamma$

- $\Gamma \leq \mathbb{R}^n$ lattice of full rank
- $\Gamma^* = \{\gamma^* \in \mathbb{R}^n : \langle \gamma^*, \gamma \rangle \in \mathbb{Z},$ for all $\gamma \in \Gamma\}$ the dual lattice of Γ

Spectral data for $\mathbb{T}^n$:

- Laplace eigenvalues: $\lambda = 4\pi\|\gamma^*\|^2$, with $\gamma^* \in \Gamma^*$
- (real) Laplace eigenfunctions: $\cos(2\pi\langle\gamma^*, \cdot\rangle)$, $\sin(2\pi\langle\gamma^*, \cdot\rangle)$
- multiplicity: $m_\Delta(\lambda)$ number of dual lattice points on circle of radius λ

Introduction

Example (Round Sphere)

Let $\mathbb{S}^n$ be the round sphere.

- $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$
- $\mathscr{P}_k(\mathbb{R}^{n+1})$ harmonic homogeneous polynomials of degree k on $\mathbb{R}^{n+1}$

Spectral Data

- Laplace eigenvalues: $\widetilde{\lambda}_k = k(k+n-1)$
- (real) Laplace eigenfunctions: $\phi = P \restriction \mathbb{S}^n$ for $P \in \mathscr{P}_k(\mathbb{R}^{n+1})$
- multiplicity: $m_\Delta(\widetilde{\lambda}_k) = \binom{n+k}{k} - \binom{n+k-1}{k-1}$

Introduction

Example (Compact Symmetric Spaces)

Let G/K be a compact irreducible symmetric space

- Let $\widehat{G}_K$ denote the irreducible representations of G over $\mathbb{R}$ with non-trivial K-fixed vectors
- ρ half the sum of the positive weights

Spectral Data

- Laplace eigenvalues: $c(\mu) = \|\mu + \rho\| - \|\rho\|$, where $V_\mu \in \widehat{G}_K$ has highest weight μ.
- Laplace eigenfunctions: matrix coefficients of representations in $\widehat{G}_K$.
- multiplicity: $m_\Delta(\lambda) = \sum_{c(\mu)=\lambda} \dim V_\mu$, where $V_\mu \in \widehat{G}_K$ irreducible of highest weight μ

Basic Questions Cont'd. Unfortunately, it appears that it is only possible to know the Laplace spectrum and/or Laplace eigenfunctions explicitly in very special cases. In particular, Riemannian manifolds that admit non-trivial isometry groups are sometimes amenable to explicit computation.

- Nevertheless, is it possible to understand the generic or "typical" situation with regard to Laplace eigenvalues and eigenfunctions?

Introduction

Definition

Let M be a closed Riemannian manifold. A Riemannian metric $g \in \mathscr{R}(M)$ is said to have **simple** Laplace spectrum, if the Δ_g-eigenspaces are one-dimensional.

Definition

Let X be a topological space.

1. $\mathscr{Y} \subseteq X$ is said to be a **residual set** if there is a countable collection $\{Y_j\}_{j=1}^{\infty}$ of open and dense subsets so that

$$\mathscr{Y} = \cap_{j=1}^{\infty} Y_j.$$

2. A property $\mathcal{P}$ is said to be **generic** in X, if the collection of points in X that enjoy property $\mathcal{P}$ contains a residual set of X.

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Introduction

Theorem (Uhlenbeck, 1976)

Let M be a closed manifold of dimension at least two and, for each positive integer $\ell \geq 2$, $\mathscr{R}^{\ell}(M)$ is the space of C^{ℓ}-Riemannian metrics on M. There is a residual set $\mathscr{Y} \subset \mathscr{R}^{\ell}(M)$ such that for any $g \in \mathscr{Y}$:

1. *Δ_g has simple spectrum, and*
2. *zero is a regular value of any Δ_g-eigenfunction.*

That is, properties (1) and (2) are generic in $\mathscr{R}^{\ell}(M)$.

Remark

Uhlenbeck's framework can be shown to apply to other families of second order elliptic operators. For example, fix a Riemannian metric on a compact manifold M and consider the Schrödinger operator $\Delta_g + b$, where $b \in C^k(M)$. Then, $\Delta_g + b$ has simple spectrum for a generic potential function $b \in C^k(M)$.

Outline

Symmetry and Generic Irreducibility of Eigenspaces

Uhlenbeck Statement 1: For a generic metric $g \in \mathscr{R}^{\ell}(M)$, where $\ell \geq 2$, the Δ_g-eigenspaces are one-dimensional.

In contrast, as we saw previously, flat tori, round spheres and, more generally, compact symmetric spaces do not have simple spectrum.

This is a consequence of symmetry.

Symmetry and Generic Irreducibility of Eigenspaces

- (regular action) For $F \in \mathrm{Diff}(M)$ and $\phi \in H^k(M;\mathbb{R})$, we have

$$\rho_F(\phi) \equiv \phi \circ F^{-1}.$$

- (Criterion for Isometry) Let $F \in \mathrm{Diff}(M)$, then $F \in \mathrm{Isom}(M,g)$ if and only if, for any $\phi \in H^k(M;\mathbb{R})$, we have

$$\Delta_g \circ \rho_F(\phi) = \rho_F(\Delta_g \phi).$$

- (Eigenspaces are preserved) $\Delta_g \phi = \lambda \phi$, then

$$\Delta_g \rho_F(\phi) = \lambda \rho_F(\phi).$$

- G acts freely on (M,g) via isometries, then Δ_g-eigenspaces are representations of G under the regular representation.

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Symmetry and Generic Irreducibility of Eigenspaces

Proposition (Donnelly, 1978)

Let G be a compact group acting freely on M with $\dim M/G \geq 1$. Then, for any G-invariant metric g, there is a Δ_g-eigenspace of dimension at least two and, in the event G is semi-simple, there are Δ_g-eigenspaces of arbitrarily large dimension.

Moral

In the presence of a symmetry group G, the best analogue of Uhlenbeck's result one can hope for is that the eigenspaces are irreducible representations of G.

Symmetry and Generic Irreducibility of Eigenspaces

Definition

Let G be a compact Lie group or finite group acting freely on a closed manifold M and let $\mathscr{R}_G(M)$ denote the space of G-invariant metrics. A metric $g \in \mathscr{R}_G(M)$ is said to have ***G*-simple** Laplace spectrum, if each real Δ_g-eigenspace is an irreducible representation of G.

Problem

Let G be a compact or finite group acting freely and effectively on a closed manifold M with $\dim M \geq 2$. Does a generic G-invariant metric on M have G-simple spectrum?

Symmetry and Generic Irreducibility of Eigenspaces

- (Zelditch, 1989) $p : M \to N$ be a finite normal cover with covering group G. Assume, "high dimension low degree (HDLD)"; i.e., $\dim M$ is greater than the dimension/degree of any irreducible representation of G. Then, a generic G-invariant C^∞-metric on M has G-simple spectrum.
- (Yau, 1993) Question appears as Problem 42 in Yau's "Open Problems in Geometry".
- (Schueth, 2017) Let $G = \underbrace{SU(2) \times \cdots \times SU(2)}_{k-\text{times}} \times \mathbb{T}^d$, then a generic left-invariant metric on G has G-simple spectrum.
- (Gomez and Marrocos, 2019) Generic irreducibility for trivial SU(2)-principal bundles
- (Jung-Zelditch, 2020) Let $M \to \Sigma$ be a principal S^1-bundle over a surface. Then, a generic S^1-invariant (Kaluza-Klein) metric on M has S^1-simple spectrum.

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Symmetry and Generic Irreducibility of Eigenspaces

Theorem (Cianci, Judge, Lin and S–)

Let $\mathbb{T} = (\mathbb{R}/2\pi\mathbb{Z})^d$ be a non-trivial torus acting freely on a connected smooth manifold M with $\dim M > \dim \mathbb{T}$ and, let $\mathscr{R}^\ell_{\mathbb{T}}(M)$ denote the space of C^ℓ $\mathbb{T}$-invariant metrics on M. Then, for each $\ell \geq 2$, there exists a residual subset $\mathscr{Y} \subseteq \mathscr{R}^\ell_{\mathbb{T}}(M)$ such that if $g \in \mathscr{Y}$, then g has $\mathbb{T}$-simple Laplace spectrum.

Remark

This result and the previous are mathematically rigorous instances of the belief in physics that the eigenspaces of the typical Hamiltonian should be irreducible. For physicists, an eigenspace that is not a G-irreducible is considered to be an "accidental degeneracy."; for example, see (Wigner, 1959). To what degree are "accidental degeneracies" an artifact of "hidden symmetries"?

Symmetry and Generic Irreducibility of Eigenspaces

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Symmetry and Generic Irreducibility of Eigenspaces

Example (Hopf Bundles)

Each odd-dimensional sphere S^{2n+1} is the total space of a Hopf bundle:

$$S^1 \hookrightarrow M \equiv S^{2n+1} \to B \equiv \mathbb{C}P^n.$$

Therefore, our theorem shows that a generic S^1-invariant metric on S^{2n+1} has S^1-simple Laplace spectrum.

Example (Two-Step Nilmanifolds)

Every closed two-step nilmanifold M is a principal torus-bundle over a torus; that is, there is a torus $\mathbb{T}$ acting freely on M such that $M/\mathbb{T}$ is a torus. And, by a result of Palais and Stewart (1960), all principal torus bundles over a torus are two-step nilmanifolds. Therefore, our result shows that a generic $\mathbb{T}$-invariant metric on a two-step nilmanifold M has $\mathbb{T}$-simple Laplace spectrum.

Symmetry and Generic Irreducibility of Eigenspaces

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Symmetry, Nodal Domains & Nodal Sets

Uhlenbeck Statement 2: For a generic metric $g \in \mathscr{R}^\ell(M)$ a Laplace eigenfunction ϕ has the property that $\phi^{-1}(0)$ is an $(n-1)$-dimensional hypersurface (possibly with many connected components)

Definition

Let ϕ be a Laplace eigenfunction on the Riemannian manifold (M, g).

1. The **nodal set** of ϕ is $\mathcal{Z}_\phi \equiv \phi^{-1}(0)$.
2. A **nodal domain** of ϕ is a connected component of $M \backslash \mathcal{Z}_\phi$. And, we let $\mathcal{N}(\phi)$ denote the number of nodal domains of ϕ.

Symmetry, Nodal Domains & Nodal Sets

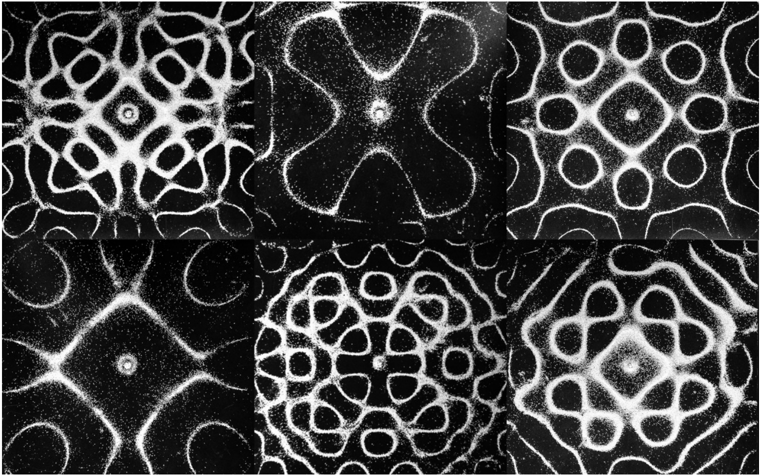

Figure: Six Chaldni plates. (http://dataphys.org/list/chladni-plates/)

Symmetry, Nodal Domains & Nodal Sets

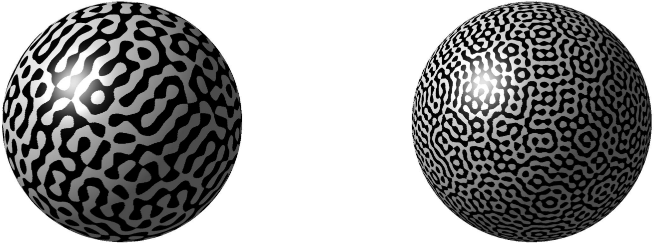

Figure: Nodal domains on $\mathbb{S}^2$ associated to a spherical harmonic of degree 40 (left) and degree 100 (right). (Image by Alex Barnett.)

Symmetry, Nodal Domains & Nodal Sets

Question

How many nodal domains can an eigenfunction have?

Symmetry, Nodal Domains & Nodal Sets

Lemma

Let (M, g) *be closed and connected Riemannian manifold and* ϕ *a non-constant* Δ_g*-eigenfunction. Then,*

$$\mathcal{N}(\phi) \geq 2.$$

Proof.

ϕ is orthogonal to the constant functions on M:

$$\int_M \phi = \int_M \phi \cdot 1 = 0.$$

Therefore, the non-constant eigenfunction ϕ must change sign and we conclude, since the sign of an eigenfunction is constant on nodal domains, that ϕ has at least two nodal domains. □

Symmetry, Nodal Domains & Nodal Sets

Theorem (Courant's Nodal Domain Theorem, 1923 & 1976)

Let (M, g) be a closed Riemannian manifold and $\{\phi_k\}$ an orthonormal basis of Δ_g-eigenfunctions with $\Delta_g \phi_k = \lambda_k \phi_k$. If ϕ is such that $\Delta_g \phi = \lambda \phi$, then we find:

$$\mathcal{N}(\phi) \leq \min\{k + 1 : \lambda_k = \lambda\}.$$

In particular, $\mathcal{N}(\phi_k) \leq k + 1$, for each $k \geq 0$.

Remark

Courant (1923) proved this in dimension two. S-Y Cheng (1976) generalized it to arbitrary compact manifolds.

Symmetry, Nodal Domains & Nodal Sets

Definition

1. An eigenvalue λ is said to be **Courant sharp**, if it has an eigenfunction ϕ such that
$$\mathcal{N}(\phi) = \min\{k+1 : \lambda_k = \lambda\}.$$
2. We will call a sequence $\langle \Psi_j \rangle_{j=1}^{\infty}$ of orthogonal Δ_g-eigenfunctions is a
   - **Courant sequence**, if $\lim_{j\to\infty} \mathcal{N}(\Psi_j) = +\infty$.
   - **nodally minimal sequence**, if $\mathcal{N}(\Psi_j) = 2$, for all j.
   - **nodally bounded sequence**, if $\mathcal{N}(\Psi_j) \leq N$, for all j and some $N \in \mathbb{N}$.

Note: "Courant Sequence", "nodally minimal sequence" and "nodally bounded sequence" are not standard terminology. Just using them for this talk!

Symmetry, Nodal Domains & Nodal Sets

Problem

Let (M, g) be a closed Riemannian manifold

1. To what extent can we find sequences of eigenfunctions that are "courant sharp", "courant", "nodally minimal", or "nodally bounded"?
2. Can we improve upon the "minimum growth rate" of 2 for $\mathcal{N}(\phi_\lambda)$ as $\lambda \to \infty$? If not, how common are eigenfunctions that possess exactly two nodal domains?
3. How do geometric and dynamical characteristics of (M, g) influence our answers?

Symmetry, Nodal Domains & Nodal Sets

Theorem (Pleijel(1956), Bérard-Meyer (1984))

There is a constant $0 < C = C(n) < 1$ *such that*

$$0 \leq \limsup_{k \to \infty} \frac{\mathcal{N}(\phi_k)}{k} \leq C.$$

That is, equality in Courant's theorem can only hold for finitely many k*:*

$$\limsup_{k \to \infty} \frac{\mathcal{N}(\phi_k)}{k} \leq \left(\frac{2}{j_1}\right)^2 \approx 0.691,$$

where j_1 *is the first zero of the Bessel function* $J_0(x)$*.*

Example (Uniformly Bounded $\mathcal{N}(\phi)$)

On the following spaces one can find an orthonormal basis $\langle \phi_j \rangle$ of Laplace eigenfunctions that admits an explicit proper subsequence $\langle \phi_{j_k} \rangle$ for which $\mathcal{N}(\phi_{j_k})$ is uniformly bounded:

- (Stern, 1925) the square
- (Lewy, 1977) the round sphere $\mathbb{S}^2$

Moral: There are no non-trivial universal lower bounds on $\mathcal{N}(\phi_\lambda)$ as $\lambda \to +\infty$.

Example (A Closer Look at $\mathbb{S}^2$: a Result of Lewy)

Lewy (1977) constructs an explicit sequence $\langle \Psi_j \rangle_{j=1}^{\infty}$ of spherical harmonics of degree j such that $\mathcal{N}(\Psi_{2k}) = 3$ and $\mathcal{N}(\Psi_{2k+1}) = 2$ for all k. The eigenfunctions $\{\Psi_k\}$ span a subspace of Weyl density zero.

For each $x > 0$ let $E_x \leq L^2(M)$ denote the direct sum of Δ_g-eigenspaces corresponding to eigenvalues less tham x. Then, the **Weyl density (with respect to Δ_g)** of a subspace $V \leq L^2(M)$ is defined to be:

$$\limsup_{x \to \infty} \frac{\dim V \cap E_x}{\dim E_x}.$$

Moral: There are no non-trivial universal lower bounds on $\mathcal{N}(\phi_\lambda)$ as $\lambda \to +\infty$. In fact, one can find a nodally minimal sequence of eigenfunctions that span a subspace of Weyl density zero.

Symmetry, Nodal Domains & Nodal Sets

Example (A Closer Look at $\mathbb{S}^2$: a Result of Nazarov-Sodin)

Let f be a random Gaussian spherical harmonic of degree k. Nazarov-Sodin (2009) showed that, as k tends to infinity, the mean of $\mathcal{N}(f)/k^2$ converges to a positive constant C and $\mathcal{N}(f)/k^2$ exponentially concentrates around C; that is, for any $\epsilon > 0$, we have

$$\mathbb{P}\left\{\left|\frac{\mathcal{N}(f)}{k^2} - C\right| > \epsilon\right\} \leq a(\epsilon)e^{-b(\epsilon)k}.$$

Moral: The number of nodal domains of "typical" eigenfunctions on $\mathbb{S}^2$ will go to infinity as the energy level increases.

Symmetry, Nodal Domains & Nodal Sets

- (Blum-Gnutzmann-Smilansky, 2002) Evidence that, in dimension two, as $\lambda \to +\infty$ the distribution of the nodal count is
  1. universal for integrable (resp., chaotic) geodesic flows and
  2. clearly distinguishes between integrable and chaotic systems.
- Conjecture (Bogomolny-Schmit, 2002): Let (Σ, g) be a surface with ergodic geodesic flow, then

$$\lim_{j\to\infty} \frac{\mathcal{N}(\phi_j)}{j} = \frac{3\sqrt{3}-5}{\pi} \approx 0.0624373...$$

Moral: (For a surface) ergodic geodesic flow implies the number of nodal domains will grow.

Symmetry, Nodal Domains & Nodal Sets

- (Jung-Zelditch, 2016) Let (Σ, J) be a Riemann surface of genus at least two and σ and anti-holomorphic involution such that $\mathrm{Fix}(\sigma)$ divides Σ into two components. Then, for a generic σ-invariant metric on Σ, every orthonormal basis contains a Courant subsequence that spans a subspace of Weyl density one.
- (Jang-Jung, 2018) if (Σ, h) is a surface possessing an orientation-reversing isometry τ such that $\mathrm{Fix}(\tau)$ separates Σ and $\langle \phi_j \rangle$ is an orthonormal basis of Δ_h-eigenfunctions for which Quantum Unique Ergodicity (QUE) holds, then $\lim_{j \to \infty} \mathcal{N}(\phi_j) = +\infty$.

Symmetry, Nodal Domains & Nodal Sets

Question

- What about spaces of higher dimension ≥ 3?
- What about spaces without ergodic/chaotic geodesic flows?

Symmetry, Nodal Domains & Nodal Sets

Theorem (Cianci, Judge, Lin, S–)

Let $\mathbb{T} = (\mathbb{R}/2\pi\mathbb{Z})^d$ be a d-dimensional torus acting freely on an n-dimensional closed manifold M with quotient $B = M/\mathbb{T}$, and fix $\ell \geq 2$.

1. *If $\dim B \geq 1$, there is a residual set $\mathscr{Y}_1 \subset \mathcal{M}_\ell^{\mathbb{T}}$ such that, for any $g \in \mathscr{Y}_1$ and Δ_g-eigenfunction ϕ, we have $\mathcal{Z}_\phi \equiv \phi^{-1}(0)$ is a smooth hypersurface.*
2. *If $\dim B \geq 2$ and, for each $\alpha \neq 0 \in \mathbb{Z}^d$, $M^\alpha \to B$ (a natural associated circle bundle) is non-trivial, then there is a residual set $\mathcal{Y}_2 \subset \mathcal{M}_\ell^{\mathbb{T}}$ such that, for any non-invariant Δ_g-eigenfunction ϕ, $\mathcal{Z}_\phi$ is a smooth connected hypersurface and $\mathcal{N}(\phi) = 2$. Therefore, for $g \in \mathscr{Y}_2$, there is a subspace of $L^2(M)$ of Weyl-Density one (with respect to Δ_g) that is spanned by Δ_g-eigenfunctions possessing exactly two nodal domains.*

Symmetry, Nodal Domains & Nodal Sets

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Forming the circle bundles $M^\alpha \to B$:

- The non-trivial irreducible representations of $\mathbb{T} = (\mathbb{R}/2\pi\mathbb{Z})^d$ are given by

$$\tau_\alpha : \Theta \in \mathbb{T} \longmapsto R_{\alpha\cdot\Theta} \equiv \begin{pmatrix} \cos(\alpha\cdot\Theta) & -\sin(\alpha\cdot\Theta) \\ \sin(\alpha\cdot\Theta) & \cos(\alpha\cdot\Theta) \end{pmatrix} \in SO(2)$$

  where $\alpha \neq 0 \in \mathbb{Z}^d$.
- For $\alpha \neq 0 \in \mathbb{Z}^d$, consider the vector bundle

$$E^\alpha = \left(M \times \mathbb{R}^2\right)/\mathbb{T} \to B = M/\mathbb{T},$$

  where $\mathbb{T}$ acts on $\mathbb{R}^2$ via τ_α
- Form the oriented circle bundle

$$M^\alpha \to M/\mathbb{T}$$

  by removing the zero section and quotienting each fiber by $\mathbb{R}_+$.

Symmetry, Nodal Domains & Nodal Sets

Corollary (Cianci, Judge, Lin and S–)

Suppose B is a connected closed manifold for which $H^2(B;\mathbb{Z})$ contains no non-trivial element of finite order. Let M be a total space of a non-trivial oriented S^1-bundle $M \to B$. Then, for any generic S^1-invariant metric g on M, each equivariant, yet non-invariant eigenfunction of Δ_g has exactly two nodal domains.

Proof.

- Euler class $e(N) \in H^2(B;\mathbb{Z})$ is a complete invariant of a circle bundle $N \to B$.
- $e(M) \neq 0$, since $M \to B$ is non-trivial.
- $e(M^\alpha) = \alpha e(M) \neq 0$, for any $\alpha \neq 0 \in \mathbb{Z}$
- Therefore, $M^\alpha \to B$ is non-trivial, for any $\alpha \neq 0 \in \mathbb{Z}$
- Apply theorem

□

Example (Hopf Bundles Revisited)

- Each odd-dimensional sphere S^{2n+1} is the total space of a Hopf bundle:

$$S^1 \hookrightarrow M \equiv S^{2n+1} \to B \equiv \mathbb{C}P^n.$$

- $M \to B$ is non-trivial
- Therefore, $M^\alpha \to B$ is non-trivial for each $\alpha \in \mathbb{Z}$
- Conclude, any Δ_g-eigenbasis associated to a generic circle-invariant metric g on S^{2n+1} has a density-one sequence consisting of equivariant, yet non-invariant functions that have precisely two nodal domains.

Moral: This stands in stark contrast to the round sphere.

Remarks

- Our theorems generalize results of Jung-Zelditch (2020) regarding free S^1-actions on three manifolds and completely answer Yau's Problem 42 for free torus actions.
- Along with Jung-Zelditch (2020), these are the first examples of Riemannian manifolds for which there is a uniform bound on $\mathcal{N}(\phi)$ for "typical" Laplace eigenfunctions. Moreover, the number of nodal domains associated to each "typical" eigenfunction has been computed precisely to be two, the smallest possible number.
- Consistent with the idea that ergodicity might play a (significant) role in the presence and abundance of density-one Courant sequences (in dimension ≥ 3).

Outline

Proof Strategy

A. Decompose $H^k(M;\mathbb{R})$ into $\mathbb{T}$ isotypical components

- Let $\mathbb{T} = (\mathbb{R}/2\pi\mathbb{Z})^d$ be a d-dimensional torus acting freely on M with $\dim M = n > \dim \mathbb{T} = d$
- The non-trivial irreducible representations of $\mathbb{T}$ are:

$$\tau_\alpha : \Theta \in \mathbb{T} \longmapsto R_{\alpha\cdot\Theta} \equiv \begin{pmatrix} \cos(\alpha\cdot\Theta) & -\sin(\alpha\cdot\Theta) \\ \sin(\alpha\cdot\Theta) & \cos(\alpha\cdot\Theta) \end{pmatrix} \in \mathrm{SO}(2)$$

  where $\alpha \neq 0 \in \mathbb{Z}^d$.
- We will denote the irreducible representations of $\mathbb{T}$ by

$$(\tau_\alpha, V_\alpha),$$

  where $V_\alpha = \mathbb{R}$ for $\alpha = 0$ and $V_\alpha = \mathbb{R}^2$ for $\alpha \neq 0 \in \mathbb{Z}^d$.
- Fix $F \subset \mathbb{Z}^d$ such that $F \cap -F = \{0\}$ and $\mathbb{Z}^d = F \cup -F$.

Proof Strategy

- $\mathbb{T}$ acts on $H^k(M;\mathbb{R})$ via the regular representation

$$\rho_\Theta(f)(x) \equiv f(\Theta^{-1}\cdot x), \text{ where } \Theta \in \mathbb{T}, f \in H^k(M;\mathbb{R}).$$

- Decompose the regular representation into its isotypical components:

$$H^k(M;\mathbb{R}) = \bigoplus_{\alpha\in F} \underbrace{H^k(M;\mathbb{R})_\alpha}_{\substack{\text{generated by} \\ \text{copies of } (\tau_\alpha, V_\alpha)}} .$$

$H^k(M;\mathbb{R})_\alpha \simeq \mathrm{Hom}_{\mathbb{T}}(V_\alpha, H^k(M;\mathbb{R})) \otimes_{D_\alpha} V_\alpha$, where $D_\alpha = \mathrm{Hom}_{\mathbb{T}}(V_\alpha, V_\alpha) \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$.

Proof Strategy

B. The Laplacian preserves isotypical components

- $\mathscr{R}^{\ell}_{\mathbb{T}}(M)$ denotes the space of $\mathbb{T}$-invariant C^{ℓ}-Riemannian metrics
- for any $g \in \mathscr{R}^{\ell}_{\mathbb{T}}(M)$, the regular representation commutes with Δ_g:

$$(\Delta_g \circ \rho_\Theta)f = (\rho_\Theta \circ \Delta_g)f.$$

- Therefore, for any $\alpha \in \mathbb{Z}^d$, $\Delta_g : H^k(M;\mathbb{R})_\alpha \to H^{k-2}(M;\mathbb{R})_\alpha$ and, letting $\Delta_{g,\alpha} \equiv \Delta_g \restriction H^k(M;\mathbb{R})_\alpha$, we have $Spec(\Delta_{g,\alpha})$

$$0 \leq \lambda_1^\alpha \leq \lambda_2^\alpha \leq \cdots \leq \lambda_k^\alpha \nearrow +\infty.$$

- We conclude $\mathrm{Spec}(\Delta_g) = \cup_{\alpha \in \mathbb{Z}^d} \mathrm{Spec}(\Delta_{g,\alpha})$.

$$L^2(M;\mathbb{R}) = \left(\bigoplus_{\lambda \in \mathrm{Spec}(\Delta_{g,0})} \underbrace{L^2(M;\mathbb{R})_{0,\lambda}}_{\substack{\mathbb{T}-\text{invariant} \\ \text{eigenfunctions}}} \right) \bigoplus \left(\bigoplus_{\alpha \neq 0 \in F} \left(\bigoplus_{\lambda \in \mathrm{Spec}(\Delta_{g,\alpha})} \underbrace{L^2(M;\mathbb{R})_{\alpha,\lambda}}_{\substack{\mathbb{T}-\text{equivariant} \\ \text{yet non-invariant} \\ \text{eigenfunctions}}} \right) \right)$$

Proof Strategy

C. Strategy (Version 1) We can establish generic $\mathbb{T}$-simple spectrum by establishing the following.

- **Generic Spectral Separation of Isotypical Components:** There is a residual set $\mathcal{Y}_1 \subset \mathscr{R}^\ell_{\mathbb{T}}(M)$ such that for any $g \in \mathscr{Y}_1$ and $\alpha \neq \beta \in F$, $\Delta_{g,\alpha}$ and $\Delta_{g,\beta}$ have no common eigenvalues.
- **Generic $\mathbb{T}$-simple spectrum on Isotypical Components:** There is a residual set $\mathcal{Y}_2 \subset \mathscr{R}^\ell_{\mathbb{T}}(M)$ such that for any $g \in \mathscr{Y}_2$ and $\alpha \in \mathbb{Z}^d$, $\Delta_{g,\alpha}$ has $\mathbb{T}$-simple spectrum.
- Taking $\mathscr{Y} = \mathscr{Y}_1 \cap \mathscr{Y}_2$ establishes the result.

Remark

If we had one metric $g_0 \in \mathscr{R}_{\mathbb{T}}(M)$ with $\mathbb{T}$-simple spectrum, then it would follow that a generic metric $g \in \mathscr{R}_{\mathbb{T}}(M)$ has $\mathbb{T}$-simple spectrum.

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Proof Strategy

D. Interlude: In principal, we only need one metric with $\mathbb{T}$-simple spectrum!!

Suppose $g_0 \in \mathscr{R}_{\mathbb{T}}(M)$ has $\mathbb{T}$-simple spectrum

1. Generic spectral separation of isotypical components
   - For $\alpha \neq \pm\beta \in \mathbb{Z}^d$ define

$$\mathcal{D}_{\alpha,\beta}(\Lambda) = \{g \in \mathscr{R}_{\mathbb{T}}(M) : \mathrm{Spec}(\Delta_{g,\alpha}) \cap [0,\Lambda] \cap \mathrm{Spec}(\Delta_{g,\beta}) = \emptyset\}$$

   - $\mathcal{D}_{\alpha,\beta}(\Lambda)$ is open for all $\Lambda > 0$ (don't need g_0 for this)
   - $\mathcal{D}_{\alpha,\beta}(\Lambda)$ is dense
     - For $g \in \mathscr{R}_{\mathbb{T}}(M)$, consider the analytic path $g_t = (1-t)g_0 + tg$
     - For $j,k \in \mathbb{N}$, $\lambda_k^{\alpha}(t)$ and $\lambda_j^{\beta}(t)$ are analytic
     - $g_t \notin \mathcal{D}_{\alpha,\beta}(\Lambda)$ for countably many t
   - Take $\mathscr{Y}_1 = \cap_{\alpha\neq\pm\beta} \cap_{N\in\mathbb{N}} \mathcal{D}_{\alpha,\beta}(N)$
2. Generic $\mathbb{T}$-simple spectrum on isotypical components
   - Use analyticity of eigen-branches.

But, it is unclear how to explicitly construct a $\mathbb{T}$-invariant metric with $\mathbb{T}$-simple Laplace spectrum. So, we need another approach.

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Proof Strategy

E. Perturbations in $\mathscr{R}^{\ell}_{\mathbb{T}}(M)$

- Let $B = M/\mathbb{T}$,
- we have the following bijection

$$(h, k, \omega) \in \mathscr{R}^{\ell}(B) \times C^{\ell}(M; \mathrm{Inn}(\mathfrak{t}))^{\mathbb{T}} \times \Omega^1_{\mathrm{conn}}(M) \mapsto g_{(h,k,\omega)} \in \mathscr{R}^{\ell}_{\mathbb{T}}(M),$$

  where
  - (Metrics on Base) $\mathscr{R}^{\ell}(B)$ is the space of C^{ℓ}-metrics on the base
  - (Metrics along Fibers) $\mathrm{Inn}(\mathfrak{t})$ is the space of inner products on $\mathfrak{t}$, the Lie algebra of $\mathbb{T}$, so that $C^{\ell}(M; \mathrm{Inn}(\mathfrak{t}))^{\mathbb{T}}$ is the space of C^{ℓ} metrics on the fibers
  - (Horizontal Distribution) $\Omega^1_{\mathrm{conn}}(M)$ is the space of connection one-forms on M
- Therefore, we have three natural types of perturbations at our disposal in applying the Uhlenbeck framework

Proof Strategy

F. Change of Viewpoint: It is advantageous to consider vector-valued functions, rather than scalar-valued

- Recall that for each $\alpha \in \mathbb{Z}^d$, we have
- $H^k(M; V_\alpha)$ collection of $\mathbf{u} = (u_1, u_2)^t : M \to V_\alpha$ such that
  - $u_1, u_2 \in H^k(M; \mathbb{R})$
  - $\tau_\alpha(\Theta)\mathbf{u}(x) = \mathbf{u}(\Theta^{-1}x)$
- For each $\alpha \in \mathbb{Z}^d$ we have an isomorphism $\Psi_\alpha : H^k(M; V_\alpha) \to H^k(M; \mathbb{R})_\alpha$

$$\mathbf{u} \mapsto u_1 .$$

- Define $\widetilde{\Delta}_{g,\alpha} : H^k(M; V_\alpha) \to H^{k-2}(M; V_\alpha)$ by

$$\widetilde{\Delta}_{g,\alpha}\mathbf{u} = (\Delta_{g,\alpha}u_1, \Delta_{g,\alpha}u_2)$$

- $\Psi_\alpha \circ \widetilde{\Delta}_{g,\alpha} = \Delta_{g,\alpha} \circ \Psi_\alpha$.

Proof Strategy

G. Strategy (Version 2) We can establish generic $\mathbb{T}$-simple spectrum by establishing the following.

- **Generic Spectral Separation of Isotypical Components:** There is a residual set $\mathcal{Y}_1 \subset \mathscr{R}^{\ell}_{\mathbb{T}}(M)$ such that for any $g \in \mathscr{Y}_1$ and $\alpha \neq \beta \in F$, $\widetilde{\Delta}_{g,\alpha}$ and $\widetilde{\Delta}_{g,\beta}$ have no common eigenvalues.
  - Prove by using (vertical) perturbations of metrics to show the sets $\mathcal{D}_{\alpha,\beta}(\Lambda)$, defined in Item D "Interlude", are non-empty for $\alpha \neq \beta \in F$.
- **Generic $\mathbb{T}$-simple spectrum on Isotypical Components:** There is a residual set $\mathcal{Y}_2 \subset \mathscr{R}^{\ell}_{\mathbb{T}}(M)$ such that for any $g \in \mathscr{Y}_2$ and $\alpha \in \mathbb{Z}^d$, $\widetilde{\Delta}_{g,\alpha}$ has $\mathbb{T}$-simple spectrum.
  - Prove using a version of parametric transversality and perturbations of metrics
- Set $\mathscr{Y} = \mathscr{Y}_1 \cap \mathscr{Y}_2$.

In the remaining time we will provide a little more insight into proving **Generic $\mathbb{T}$-simple spectrum on Isotypical Components**.

Proof Strategy

H. Generic $\mathbb{T}$-simple spectrum on Isotypical Components

- We employ the Uhlenbeck framework (with some modifications)
- Fix $\alpha \neq 0 \in F$.
- For $g \in \mathscr{R}^{\ell}_{\mathbb{T}}(M)$, define $F_g : H^k(M; V_\alpha) \times \mathbb{R}^2 \to H^k(M; V_\alpha)$ via

$$F_g(\mathbf{u}, a, b) = \widetilde{\Delta}_{g,\alpha}\mathbf{u} - a\mathbf{u} - b\mathbf{u}^*,$$

  where $\mathbf{u}^* = \mathscr{J}(\mathbf{u})$ and $\mathscr{J}$ is the $\mathbb{T}$-invariant complex structure on V_α.
- One can check that $F_g^{-1}(0) = \{(\mathbf{u}, \lambda, 0) : \left(\widetilde{\Delta}_{g,\alpha} - \lambda I\right)\mathbf{u} = 0\}$
- Establish the following

Lemma

Fix $\alpha \neq 0$ and let $\mathbf{u} \neq 0$ belong to $\ker(\Delta_{g,\alpha} - \lambda I)$. Then, the dimension of $\ker(\Delta_{g,\alpha} - \lambda I)$ equals two if and only if $(dF_g)_{(\mathbf{u},\lambda,0)}$ is surjective.

Proof Strategy

H. Generic $\mathbb{T}$-simple spectrum on Isotypical Components (cont'd)

- Consider $F : H^k(M; V_\alpha) \times \mathbb{R}^2 \times \mathscr{R}^\ell_{\mathbb{T}}(M) \to H^{k-2}(M; V_\alpha)$ via

$$F(\mathbf{u}, a, b, g) = F_g(\mathbf{u}, a, b) = \widetilde{\Delta}_{g,\alpha}\mathbf{u} - a\mathbf{u} - b\mathbf{u}^*,$$

- Define a natural projection

$$\pi : H^k(M; V_\alpha) \times \mathbb{R}^2 \times \mathscr{R}^\ell_{\mathbb{T}}(M) \to \mathscr{R}^\ell_{\mathbb{T}}(M).$$

- Establish the following version of parametric transversality

Proposition (Parametric Transversality)

Let $\mathcal{U} \subset \mathscr{R}^\ell_{\mathbb{T}}(M)$ be an open set. If for each $x \in F^{-1}(0) \cap \pi^{-1}(\mathcal{U})$, the differential $(dF)_x$ is surjective, then there exists a residual subset of $\mathscr{Y} \subset \mathcal{U}$ so that for each $g \in \mathscr{Y}$ and each $(\mathbf{u}, \lambda, 0) \in F_g^{-1}(0)$, the differential $(dF_g)_{(\mathbf{u},\lambda,0)}$ is surjective.

Proof Strategy

H. Generic $\mathbb{T}$-simple spectrum on Isotypical Components (cont'd)

- Fix $\alpha \neq 0 \in F$,
- define a particular $\mathcal{U}_\alpha \subseteq \mathscr{R}^\ell_{\mathbb{T}}(M)$ that is open and dense
- Using various metric perturbations in $\mathscr{R}^\ell_{\mathbb{T}}(M)$, we establish that $\mathcal{U}_\alpha$ satisfies the hypotheses of our Parametric Transversality theorem
- Let $\mathscr{Y}_\alpha \subset \mathcal{U}_\alpha$ be the residual set guaranteed by Parametric Transversality
- $\mathscr{Y} = \cap \mathscr{Y}_\alpha$ is the sought after residual set in $\mathscr{R}^\ell_{\mathbb{T}}(M)$.

THANK YOU!!