Errata for "Complex Topological K-Theory"

page 5: part (ii) of Proposition 1.1.9 should be $A^*B^* = (BA)^*$.

page 48: The first sentence of Exercise 1.5 should read, "Let V and W be vector bundles over a compact Hausdorff space, and let $\phi : V \longrightarrow W$ be a bundle homomorphism."

page 55: The displayed equation in the proof of Lemma 2.2.4 should be

$$(\mathsf{S}, \mathsf{S})^{-1}(\mathsf{E}_1, \mathsf{S}\mathsf{E}_2\mathsf{S}^{-1})(\mathsf{S}, \mathsf{S}) = (\mathsf{S}^{-1}\mathsf{E}_1\mathsf{S}, \mathsf{E}_2).$$

page 56: There is a missing right parenthesis in the third line of the display of the proof of Proposition 2.2.8; that line should be

$$[\mathrm{diag}(\mathsf{E}_1, I_m, 0_m), \mathsf{S}\,\mathrm{diag}(I_{j+m}, 0_{m+n-j})\mathsf{S}^{-1}] - [I_{j+m}, I_{j+m}].$$

pages 61 and 67: This is the correct definition of $\widetilde{\phi}$ in the statement of Theorems 2.2.14 and 2.3.17:

$$\widetilde{\phi}(z) = \begin{cases} \phi(z) & \text{if } z \in X \backslash A \\ \infty & \text{if } z = \infty. \end{cases}$$

page 78: In Definition 2.6.1, the name of the compact set is missing from the last sentence: "a subset of X^+ is open if it is open in X or if it can be written $X^+ \backslash C$ for some compact subset C of X.

page 110: In Exercise 2.10, the isomorphism in the last sentence should be $\mathrm{K}^0(\mathcal{S}^n X) \cong \mathrm{K}^0(\widetilde{\mathcal{S}}^n X)$.

page 173: In Exercise 3.2, the left vertical arrow should point up, not down.

pages 190 – 192: Unfortunately, Proposition 4.3.8, the Bianchi identity, is not correct. Fortunately, the only place in the book where I use Proposition 4.3.8 is in Theorem 4.3.10, and by using an argument from page 333 of Jonathan Rosenberg's book *Algebraic K-Theory and its Applications*, the proof of that theorem is easily modified so that it does not rely on Proposition 4.3.8. There are also a couple of typos in the computation near the bottom of page 191, so I have included the entire corrected proof of Theorem 4.3.10 here:

Theorem (Chern–Weil) *Let M be a compact manifold, let n be a natural number, and suppose that E in $\mathrm{M}(n, C^\infty(M))$ is a smooth idempotent. Then for every invariant polynomial P on $\mathrm{M}(n,\mathbb{C})$:*

(i) *the differential form $P(\nabla_{\mathsf{E}}^2)$ is closed;*
(ii) *the cohomology class of $P(\nabla_{\mathsf{E}}^2)$ in $H^*_{deR}(M)$ depends only on $[\mathsf{E}]$ in $\mathrm{Idem}(C(X))$.*
(iii) *for $f : N \longrightarrow M$ smooth, we have $P(f^*(\mathsf{E})) = f^*(P(\mathsf{E}))$ in $H_{deR}(N)$.*

Proof The sum and wedge of closed forms are closed, so to prove (i), it suffices by Corollaries 4.2.11 and 4.2.12 to verify that $\mathrm{Tr}\left(\nabla_{\mathsf{E}}^{2k}\right)$ is closed for each k. Using Lemma 4.3.6 and 4.3.7 along with the linearity of the trace and the exterior derivative, we obtain the string of equalities

$$\begin{aligned} d\left(\mathrm{Tr}\left(\nabla_{\mathsf{E}}^{2k}\right)\right) &= \mathrm{Tr}\left(d\left(\nabla_{\mathsf{E}}^{2k}\right)\right) \\ &= \mathrm{Tr}\left(d\left(\mathsf{E}(d\mathsf{E})^{2k}\right)\right) \\ &= \mathrm{Tr}\left(d\left(\mathsf{E}^2(d\mathsf{E})^{2k}\right)\right) \\ &= \mathrm{Tr}\left(\left((d\mathsf{E})\mathsf{E} + \mathsf{E}(d\mathsf{E})\right)(d\mathsf{E})^{2k}\right) \\ &= \mathrm{Tr}\left(\left((I_n - \mathsf{E})(d\mathsf{E}) + \mathsf{E}(d\mathsf{E})\right)(d\mathsf{E})^{2k}\right) \\ &= \mathrm{Tr}\left((I_n - \mathsf{E})(d\mathsf{E})^{2k+1}\right) + \mathrm{Tr}\left(\mathsf{E}(d\mathsf{E})^{2k+1}\right). \end{aligned}$$

Next, Lemma 4.3.7 and the cyclicity of the trace yield

$$\begin{aligned} \mathrm{Tr}\left(\mathsf{E}(d\mathsf{E})^{2k+1}\right) &= \mathrm{Tr}\left(\mathsf{E}^2(d\mathsf{E})^{2k+1}\right) \\ &= \mathrm{Tr}\left(\mathsf{E}(d\mathsf{E})^{2k}\mathsf{E}(d\mathsf{E})\right) \\ &= \mathrm{Tr}\left(\mathsf{E}(d\mathsf{E})^{2k}(d\mathsf{E})(I_n - \mathsf{E})\right) \\ &= \mathrm{Tr}\left((I_n - \mathsf{E})\mathsf{E}(d\mathsf{E})^{2k}(d\mathsf{E})\right) \\ &= 0. \end{aligned}$$

A similar computation shows that $\operatorname{Tr}\left((I_n - \mathsf{E})(d\mathsf{E})^{2k+1}\right) = 0$ as well.
To prove (ii), first note that

$$\operatorname{Tr}(\nabla^{2k}_{\operatorname{diag}(\mathsf{E},0)}) = \operatorname{Tr}(\nabla^{2k}_{\mathsf{E}})$$

for every natural number k and every smooth idempotent E. This observation, paired with Corollaries 4.2.11 and 4.2.12, shows that for any invariant polynomial P on $\mathrm{M}(n, \mathbb{C})$, the cohomology class of $P(\nabla^2_{\mathsf{E}})$ in $H^*_{deR}(M)$ does not depend on the size of matrix we use to represent E. To complete the proof of (ii), we will show that if $\{\mathsf{E}_t\}$ is a smooth homotopy of idempotents, then the partial derivative of $P\left(\nabla^2_{\mathsf{E}_t}\right)$ with respect to t is an exact form. To simplify notation, we will suppress the t subscript, and we will denote the derivative of E with respect to t as $\dot{\mathsf{E}}$. The sum and wedge product of exact forms are exact, so from Corollary 4.2.12 we deduce that we need only show that the partial derivative of $b_k(\nabla^2_{\mathsf{E}})$ with respect to t is exact for each k.
We begin by noting three facts. First, from the equation $\mathsf{E}^2 = \mathsf{E}$, we have $\dot{\mathsf{E}}\mathsf{E} + \mathsf{E}\dot{\mathsf{E}} = \dot{\mathsf{E}}$, and thus $\mathsf{E}\dot{\mathsf{E}} = \dot{\mathsf{E}}(I_n - \mathsf{E})$. Second, if we multiply this last equation by E, we see that $\mathsf{E}\dot{\mathsf{E}}\mathsf{E} = 0$. Third, because mixed partials are equal, the partial derivative of $d\mathsf{E}$ with respect to t is $d\dot{\mathsf{E}}$.
For each k, we have

$$\begin{aligned}
\frac{\partial}{\partial t} b_k(\nabla^2_{\mathsf{E}}) &= \frac{\partial}{\partial t}\left(\operatorname{Tr}\left(\left(\mathsf{E}(d\mathsf{E})^2\right)^k\right)\right)\\
&= \frac{\partial}{\partial t}\left(\operatorname{Tr}\left(\mathsf{E}(d\mathsf{E})^{2k}\right)\right)\\
&= \operatorname{Tr}\left(\dot{\mathsf{E}}(d\mathsf{E})^{2k}\right) + \sum_{j=1}^{2k} \operatorname{Tr}\left(\mathsf{E}(d\mathsf{E})^{j-1}(d\dot{\mathsf{E}})(d\mathsf{E})^{2k-j}\right),
\end{aligned}$$

where the second line follows from k applications of Lemma 4.3.7. We also have

$$\begin{aligned}
\operatorname{Tr}\left(\dot{\mathsf{E}}(d\mathsf{E})^{2k}\right) &= \operatorname{Tr}\left(\left(\dot{\mathsf{E}}\mathsf{E} + \mathsf{E}\dot{\mathsf{E}}\right)(d\mathsf{E})^{2k}\right)\\
&= \operatorname{Tr}\left(\dot{\mathsf{E}}\mathsf{E}(d\mathsf{E})^{2k}\right) + \operatorname{Tr}\left(\mathsf{E}\dot{\mathsf{E}}(d\mathsf{E})^{2k}\right)\\
&= \operatorname{Tr}\left(\dot{\mathsf{E}}\mathsf{E}\mathsf{E}(d\mathsf{E})^{2k}\right) + \operatorname{Tr}\left(\mathsf{E}\mathsf{E}\dot{\mathsf{E}}(d\mathsf{E})^{2k}\right)\\
&= \operatorname{Tr}\left(\dot{\mathsf{E}}\mathsf{E}(d\mathsf{E})^{2k}\mathsf{E}\right) + \operatorname{Tr}\left(\mathsf{E}\dot{\mathsf{E}}(d\mathsf{E})^{2k}\mathsf{E}\right)\\
&= \operatorname{Tr}\left(\mathsf{E}\dot{\mathsf{E}}\mathsf{E}(d\mathsf{E})^{2k}\right) + \operatorname{Tr}\left(\mathsf{E}\dot{\mathsf{E}}\mathsf{E}(d\mathsf{E})^{2k}\right)\\
&= 0 + 0 = 0,
\end{aligned}$$

and thus the first term of the derivative vanishes. From the remaining terms, we obtain

$$\sum_{j=1}^{2k} \operatorname{Tr}\left(\mathsf{E}(d\mathsf{E})^{j-1}(d\dot{\mathsf{E}})(d\mathsf{E})^{2k-j}\right) = \sum_{j=1}^{2k} \operatorname{Tr}\left((d\mathsf{E})^{2k-j}\mathsf{E}(d\mathsf{E})^{j-1}(d\dot{\mathsf{E}})\right)$$

$$= \sum_{j \text{ even}} \operatorname{Tr}\left((d\mathsf{E})^{2k-j}\mathsf{E}(d\mathsf{E})^{j-1}(d\dot{\mathsf{E}})\right) + \sum_{j \text{ odd}} \operatorname{Tr}\left((d\mathsf{E})^{2k-j}\mathsf{E}(d\mathsf{E})^{j-1}(d\dot{\mathsf{E}})\right)$$

$$= \sum_{j \text{ even}} \operatorname{Tr}\left((d\mathsf{E})^{2k-1}(I_n - \mathsf{E})(d\dot{\mathsf{E}})\right) + \sum_{j \text{ odd}} \operatorname{Tr}\left((d\mathsf{E})^{2k-1}\mathsf{E}(d\dot{\mathsf{E}})\right)$$

$$= \sum_{j \text{ even}} \operatorname{Tr}\left((d\mathsf{E})^{2k-1}(d\dot{\mathsf{E}}) - (d\mathsf{E})^{2k-1}\mathsf{E}(d\dot{\mathsf{E}})\right) + \sum_{j \text{ odd}} \operatorname{Tr}\left((d\mathsf{E})^{2k-1}\mathsf{E}(d\dot{\mathsf{E}})\right)$$

$$= k \operatorname{Tr}\left((d\mathsf{E})^{2k-1}(d\dot{\mathsf{E}})\right)$$

$$= d\left(k \operatorname{Tr}\left((d\mathsf{E})^{2k-1}\dot{\mathsf{E}}\right)\right).$$

Therefore $\frac{\partial}{\partial t} b_k(\nabla_{\mathsf{E}}^2)$ is exact.
Finally, to establish (iii), note that the exterior derivative commutes with pullbacks, which implies that $\nabla_{f^*\mathsf{E}} = f^*(\nabla_{\mathsf{E}})$. Moreover, the functorial properties of f^* give us $P(\nabla_{f^*\mathsf{E}}^2) = f^*\big(P(\nabla_{\mathsf{E}}^2)\big)$ for every invariant polynomial, and thus $P(f^*\mathsf{E}) = f^*(P(\mathsf{E}))$. □

page 201: In the notes at the end of Chapter 4, my remarks about the Chern character contain an error, as Chuck Weibel pointed out in his Math Review of the book. The Chern character takes values in $H^*(M, \mathbb{Q})$, not necessarily in $H^*(M, \mathbb{Z})$, and thus is not a ring isomorphism. On a related note, my statement that the Chern character is an isomorphism up to torsion is correct, but there exist compact manifolds for which $K^*(M)$ and $H^*(M, \mathbb{Z})$ are not isomorphic. One such example is $\mathbb{RP}^4$; the group $K^0(\mathbb{RP}^4)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_4$, while $H^{even}(\mathbb{RP}^4)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$; thanks to Tom Goodwillie for providing me with this example.